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Characterization of homoclinic points bifurcating from degenerate homoclinic orbits

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Introduction

In the last decades a lot of work has been done in studying homoclinic points in periodically forced systems. Largely this work is devoted to give sufficient conditions for the existence of transversal homoclinic points. We will only mention the book by Wiggins [8] and references therein, the papers by Palmer [5] and [6] and a paper by Zeng [10]. In the same way as in [6] and [10] we consider periodic perturbations of a degenerate homoclinic orbit asymptotic to a hyperbolic fixed point of an autonomous vector field. Here "degenerate" means that the tangent spaces of the stable and unstable manifold (of the fixed point) intersect (along the homoclinic orbit) not only in the subspace spanned by the vector field. Different from the papers mentioned above we consider periodic perturbations of a family of vector fields in which the degenerate homoclinic orbit appears generically. The case where the homoclinic orbit is embedded in a smooth family of homoclinic orbits forms a codimension infinity problem. For that reason we are only interested in the case where the stable and unstable manifolds intersect only in the (degenerate) homoclinic orbit. Also we are not interested in giving only sufficient conditions for the existence of transversal homoclinic points. Rather our aim is to compute all homoclinic points in a family of periodically forced autonomous systems such as in [3]. The procedure is very similar to that in [3]. We formulate the problem of finding homoclinic points as an operator equation. Performing a Lyapunov/Schmidt reduction we gain a bifurcation equation $B(...) = 0$. But our actual aim in this paper is to characterize the homoclinic points by the dimension D of the intersection of the tangent spaces of stable and unstable manifolds of the associated Poincaré map. In particular if this dimension is zero the corresponding homoclinic point is transversal. The main result of this paper is that D is equal to the dimension of the kernel of the derivative of B (with respect to the state variables) at the corresponding solution point (see Theorem 3). Afterwards we consider the case where the dimension of the intersection of the tangent spaces of stable and unstable manifolds along the primary homoclinic orbit is two. In [3] we already proved that the parameter values for which homoclinic points in a certain section exist, form a family of Whitney umbrellas. The second crucial result of this paper is, that (under certain conditions) only a homoclinic point associated to the singularity of a Whitney umbrella is nondegenerate - see Theorem 5. This explains how a higher dimensional penetration of the corresponding tangent spaces can be created: Similar to the autonomous case a non-transverse homoclinic point arises if by changing parameters two homoclinic points merge. See [9] or [2] for the autonomous case.

More precisely we consider the differential equation

$$\dot{x} = f(x, \lambda) + \epsilon g(t, x, \lambda, \epsilon), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ are (sufficiently) smooth. Moreover $g(\cdot, x, \lambda, \epsilon)$ is $1-$periodic.
Furthermore we assume that the autonomous system

$$\dot{x} = f(x, 0) \tag{2}$$

has a homoclinic solution $\gamma(\cdot)$ asymptotic to the hyperbolic fixed point $x = 0$ of the system (2), that is $\lim_{t \to \pm\infty} \gamma(t) = 0$.

Finally we assume

$$\dim(T_{\gamma(0)}W^s(0) \cap T_{\gamma(0)}W^n(0)) = d.$$

By $T_{\gamma(0)}W^{s(u)}(0)$ we denote the tangent space of the stable (unstable) manifold $W^{s(u)}(0)$ of the fixed point $x = 0$ (of the system (2)).
Although we will focus on the case $d = 2$, we permit d to be any natural number. And we will work with arbitrary d as far as possible. For $d > 1$ we call the homoclinic solution $\gamma(\cdot)$ degenerate. The dimension m of the $\lambda-$space results from the requirement that the system $\dot{x} = f(x, \lambda)$ is universally unfolded (with respect to the occurence of the homoclinic orbit). See [9], [2] on transversality condition (T) in this paper.

The remainder of this paper is organized as follows. In section 2 we derive the bifurcation equation and prove the main theorem concerning the intersection number D. Using a normal form for the bifurcation equation we compute D in the case $d = 2$. This has been done in section 3.

1 The bifurcation equation

In [3] the set of homoclinic points in the case $d = 2$ was evaluated. There the authors restricted themselves on the determination of homoclinic points in a section transverse to $\Gamma := \{\gamma(t),\ t \in \mathbb{R}\}$. For this purpose a bifurcation equation was used providing homoclinic points in this section only. But to compute the number D (even for homoclinic points in a section) we may not ignore directions transvers to this section. For that reason we need a bifurcation equation competent for all homoclinic points in a whole neighborhood of a point at Γ.

We reduce the bifurcation equation in a somewhat different way than in [3]. However we start as in [3] by rewriting the periodically forced system (1) in an autonomous system

$$\begin{aligned} \dot{x} &= f(x, \lambda) + \epsilon g(\theta, x, \lambda, \epsilon) \\ \dot{\theta} &= 1 \end{aligned} \tag{3}$$

with phase space $\mathbb{R}^n \times S^1$.
By means of the flow $\Phi^t_{\lambda,\epsilon}$ of (3) we define the Poincaré-map

$$\Pi_{\lambda,\epsilon} : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n \times \{0\}, \quad (x, 0) \mapsto \Phi^1_{\lambda,\epsilon}(x, 0).$$

Clearly $(0, 0)$ is a hyperbolic fixed point of $\Pi_{0,0}$. Due to this hyperbolicity the mapping $\Pi_{\lambda,\epsilon}$ has for small (λ, ϵ) a (unique) hyperbolic fixed point $p_{\lambda,\epsilon}$ near $(0, 0) \in \mathbb{R}^n \times S^1$. By $W^{s(u)}_{\lambda,\epsilon}$ we denote the stable (unstable) manifold of the fixed point $p_{\lambda,\epsilon}$.
The elements different from $p_{\lambda,\epsilon}$ in the intersection $W^s_{\lambda,\epsilon} \cap W^u_{\lambda,\epsilon}$ we call homoclinic points of (1). There is a one to one relation between homoclinic points and points $x_{\lambda,\epsilon}(0)$, where $x_{\lambda,\epsilon}(\cdot)$ is a solution of (3) satisfying the limit set property

$$\alpha(x_{\lambda,\epsilon}) = w(x_{\lambda,\epsilon}) = \{\Phi^t(p_{\lambda,\epsilon}),\ t \in \mathbb{R}\}. \tag{4}$$

Now, let $q_{\lambda,\epsilon}$ be a homoclinic point associated to $\Pi_{\lambda,\epsilon}$. Then the variational equation

$$x_{k+1} = D\Pi_{\lambda,\epsilon}\left(\Pi^k_{\lambda,\epsilon}\, q_{\lambda,\epsilon}\right) x_k \tag{5}$$

has an exponential dichotomy on both $\mathbb{Z}^+$ and $\mathbb{Z}^-$ (see [4]). Due to this fact one can prove that $D(q_{\lambda,\epsilon}) := \dim\left(T_{q_{\lambda,\epsilon}} W^s_{\lambda,\epsilon} \cap T_{q_{\lambda,\epsilon}} W^u_{\lambda,\epsilon}\right)$ is equal to the number of linearly independent solutions of (5) which are bounded on $\mathbb{Z}$. The proof is completely similar to that for the time continuous case - see for instance in [7]. This number again is equal to the number of linearly independent on $\mathbb{R}$ bounded solutions of

$$\dot{h} = D_1 f(x_{\lambda,\epsilon}(t), \lambda)h + \epsilon D_2 g(t, x_{\lambda,\epsilon}(t), \lambda, \epsilon)h. \tag{6}$$

By $D_i \cdots$ we denote the partial derivative of $\cdots$ with respect to the ith variable.
Therefore the problem of finding and characterizing the homoclinic points is equivalent to the computation of solutions of (3) satisfying (4) and to determine the number of linearly independent (on $\mathbb{R}$) bounded solutions of (6). For this end we reformulate the differential equation (3) in an operator equation and perform a Liapunov/Schmidt reduction.
In search of homoclinic points we restrict ourselves on a neighborhood of $\gamma(0)$ (in $\mathbb{R}$). Indeed this is not really a restriction because each point $\gamma(\alpha)$ of the homoclinic orbit $\Gamma := \{\gamma(t) : t \in \mathbb{R}\}$ can be written as $\gamma_\alpha(0)$, where γ_α is a homoclinic solution of (2). We seek solutions of (1) in the form:

$$x(t) = \gamma(t) + z(t). \tag{7}$$

This leads to the following differential equation

$$\dot{z} = D_1 f(\gamma(t), 0)z + \tilde{F}(t, z, \lambda, \epsilon), \tag{8}$$

where $\tilde{F} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is defined by

$$\tilde{F}(t, z, \lambda, \epsilon) := f(\gamma(t) + z, \lambda) - f(\gamma(t), 0) - D_1 f(\gamma(t), 0)z + \epsilon g(t, \gamma(t) + z, \lambda, \epsilon). \tag{9}$$

Due to the hyperbolicity of the fixed point $x = 0$ a solution $x(\cdot)$ of (1) in the form (7) satisfies the limit set property (4) if and only if $|z(t)|$ is sufficiently small for large $|t|$. This allows to reformulate the differential equation (8) in an operator equation in spaces of bounded functions.

$$G(z, \lambda, \epsilon) = 0, \tag{10}$$

where

$$G : C^1_b \times \mathbb{R}^m \times \mathbb{R} \to C^o_b$$

is defined by

$$(G(z, \lambda, \epsilon))(t) := -\dot{z}(t) + D_1 f(\gamma(t), 0)z(t) + \tilde{F}(t, z(t), \lambda, \epsilon).$$

$C_b^{o(1)}$ is the Banach space of, on $\mathbb{R}$ bounded, continuous (continuously differentiable) functions (whose first derivative is still in C_b^o). These Banach spaces are endowed with the usual norms (see for instance in [3] or [5]). It is easy to see that $G(0,0,0)=0$.
Let L be the partial Fréchet derivative of G with respect to z at $(0,0,0)$, $L := D_1G(0,0,0)$. Then $(Lz)(t) = -\dot{z}(t) + D_1 f(\gamma(t),0)z(t)$. The formal adjoint $L^* \in \mathbb{L}(C_b^1, C_b^o)$ is defined by $(L^*z)(t) := \dot{z}(t) + D_1 f(\gamma(t),0)^T z(t)$. Next we perform a Lyapunov/Schmidt reduction to gain a finite dimensional bifurcation equation. For this end we write equation (10) as a system

$$
\begin{aligned}
G^1(u,y,\lambda,\epsilon) &= PG(u+y,\lambda,\epsilon) = 0 && \text{(11-a)}\\
G^2(u,y,\lambda,\epsilon) &= (I-P)G(u+y,\lambda,\epsilon) = 0 && \text{(11-b)}
\end{aligned}
$$

where P projects C_b^o on $\operatorname{im} L$ along $\ker L^*$, u is contained in $\ker L$ and y belongs to a complement Y of $\ker L$ in C_b^1. Now (11-a) can be solved for $y = y^*(u,\lambda,\epsilon)$ near $(u,y,\lambda,\epsilon) = (0,0,0,0)$. Plugging this into (11-b) we get the bifurcation equation

$$B(u,\lambda,\epsilon) := G^2(u, y^*(u,\lambda,\epsilon),\lambda,\epsilon) = 0 \tag{12}$$

That this procedure works is proved in [3]. See also references therein.
An immediate consequence of $G(0,0,0)=0$ is

$$B(0,0,0) = 0. \tag{13}$$

Differentiating the identity $G^1(u, y^*(u,\lambda,\epsilon),\lambda,\epsilon) = 0$ with respect to u yields

$$D_1 y^*(0,0,0) = 0. \tag{14}$$

This again implies

$$D_1 B(0,0,0) = 0. \tag{15}$$

Clearly any solution (u,λ,ϵ) of (12) corresponds to a solution $x_{u,\lambda,\epsilon}(\cdot)$, $x_{u,\lambda,\epsilon}(t) = \gamma(t) + u(t) + y^*(u,\lambda,\epsilon)(t)$ of (1) satisfying the limit set property (4) and vice versa. That means nothing else but there is a one to one correspondence between solutions (u,λ,c) of (12) and homoclinic points $x_{u,\lambda,\epsilon}(0)$.
The approach introduced here differs slightly from that used in [3]. This is due to the restriction of searching homoclinic points only in a neighborhood of $\gamma(0)$. So here we do not need the additional parameter α which was introduced in [3]. Doing it in this way, the restriction on a subspace of C_b^1, that seems somewhat artificial, can be omitted - see [3, Lemma 2].
Before going more fully into the matter of investigation of the solutions of (12) we will look after the intersection number $D_{u,\lambda,\epsilon} := D(x_{u,\lambda,\epsilon}(0))$.

Lemma 1 $D_{u,\lambda,\epsilon} = \dim\ker D_1 G(u + y^*(u,\lambda,\epsilon),\lambda,\epsilon)$.

Proof With $z_{u,\lambda,\epsilon} = x_{u,\lambda,\epsilon} - \gamma$, cf. (7) we get

$$(D_1G(z_{u,\lambda,\epsilon},\lambda,\epsilon)h)(t) = -\dot{h} + D_1 f(x_{u,\lambda,\epsilon}(t),\lambda)h(t) + \epsilon D_2 g(t, x_{u,\lambda,\epsilon}(t),\lambda,\epsilon)h(t).$$

Therefore $\dim\ker D_1G(u + y^*(u,\lambda,\epsilon),\lambda,\epsilon)$ is just the number of linearly independent (on $\mathbb{R}$) bounded solutions of (6). ■

Lemma 2 *Let (u,λ,ϵ) be a solution of the bifurcation equation (12) and $(z_{u,\lambda,\epsilon},\lambda,\epsilon)$ the corresponding solution of (10). Then* $\dim\ker D_1B(u,\lambda,\epsilon) = \dim\ker D_1G(z_{u,\lambda,\epsilon},\lambda,\epsilon)$.

Proof For any u,y,λ,ϵ we have $\ker D_1G(u+y,\lambda,\epsilon) = \{\bar u+\bar y$ satisfying the following system (16-a), (16-b) $\}$.

$$D_1G^1(u,y,\lambda,\epsilon)\bar u \quad + \quad D_2G^1(u,y,\lambda,\epsilon)\bar y = 0 \tag{16-a}$$
$$D_1G^2(u,y,\lambda,\epsilon)\bar u \quad + \quad D_2G^2(u,y,\lambda,\epsilon)\bar y = 0. \tag{16-b}$$

Equation (16-a) can be solved for $\bar y = \bar y^*(u,y,\lambda,\epsilon,\bar u)$ near $(u,y,\lambda,\epsilon,\bar u,\bar y) = (0,\cdots,0)$. Plugging this into (16-b) yields

$$\ker D_1G(u+y,\lambda,\epsilon) = \{\bar u + \bar y^*(u,y,\lambda,\epsilon,\bar u) \quad \text{satisfying the following equation (18)}\} \tag{17}$$

$$D_1G^2(u,y,\lambda,\epsilon)\bar u + D_2G^2(u,y,\lambda,\epsilon)\bar y^*(u,y,\lambda,\epsilon,\bar u) = 0. \tag{18}$$

Next we show that

$$\bar y^*(u,y^*(u,\lambda,\epsilon),\lambda,\epsilon),\lambda,\epsilon,\bar u) = D_1y^*(u,\lambda,\epsilon)\bar u. \tag{19}$$

This becomes clear by differentiating the identity $G^1(u,y^*(u,\lambda,\epsilon),\lambda,\epsilon) \equiv 0$ with respect to u: $D_1G^1(u,y^*(u,\lambda,\epsilon),\lambda,\epsilon)\bar u + D_2G^1(u,y+(u,\lambda,\epsilon,\epsilon),\lambda,\epsilon)D_1y^*(u,\lambda,\epsilon)\bar u = 0$, $\forall \bar u$.
Now (17) and (19) provide the lemma. ■

We summarize our results in the following theorem:

Theorem 3 *Let (u,λ,ϵ) be a solution of the bifurcation equation (12). Then* $D_{u,\lambda,\epsilon} = \dim\ker D_1B(u,\lambda,\epsilon)$. ■

2 Homoclinic points

As already mentioned we are interested in all homoclinic points in a neighborhood of $\gamma(0)$. These homoclinic points are determined by solutions of the bifurcation equation (12). Beyond we want to consider periodic perturbations of an universally unfolded autonomous system. This provides a lower estimation for m. Obviously, for $\epsilon = 0$ the homoclinic points of (1) are related to homoclinic orbits of the unperturbed system $\dot x = f(x,\lambda)$. Moreover there is a one-to-one relation between homoclinic orbits of $\dot x = f(x,\lambda)$ and homoclinic points of (1) in a section Σ at $\gamma(0)$ transverse to $\dot\gamma(0)$.
In particular let Σ be a hyperplan orthogonal to $\dot\gamma(0)$ (with respect to a scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$). Then the homoclinic points in Σ are related to solutions of the system

$$B(u,\lambda,\epsilon) = 0, \tag{20-a}$$
$$X(u,\lambda,\epsilon) := \langle (u + y^*(u,\lambda,\epsilon))(0),\ \dot\gamma(0)\rangle = 0 \tag{20-b}$$

Taking into consideration that $\dot{\gamma} \in \ker L$ we have a unique representation $u = \sigma\dot{\gamma} + v$, where v belongs to a complement Y of span $\{\dot{\gamma}\}$ in $\ker L$. This allows to rewrite (20) as

$$\tilde{B}(\sigma, v, \lambda, \epsilon) := B(\sigma\dot{\gamma} + v, \lambda, \epsilon) = 0, \tag{21-a}$$
$$\tilde{X}(\sigma, v, \lambda, \epsilon) := X(\sigma\dot{\gamma} + v, \lambda, \epsilon) = 0. \tag{21-b}$$

Using (14) we get

$$D_1\tilde{X}(0,0,0) = \langle \dot{\gamma}(0), \dot{\gamma}(0) \rangle \neq 0.$$

Therefore we can solve (21-b) for $\sigma = \sigma^*(u, \lambda, \epsilon)$ near $(\sigma, v, \lambda, \epsilon) = (0,0,0,0)$. Plugging this into (21-a) we get

$$\hat{B}(v, \lambda, \epsilon) := \tilde{B}(\sigma^*(v, \lambda, \epsilon), v, \lambda, \epsilon) = 0. \tag{22}$$

$\hat{B}$ has the singularity

$$\hat{B}(0,0,0) = 0\ , \quad D_1\hat{B}(0,0,0) = 0. \tag{23}$$

The solutions of (22) are connected with homoclinic points in Σ and especially for $\epsilon = 0$ the solutions of $\hat{B}(v, \lambda, \epsilon) = 0$ are associated with homoclinic orbits of $\dot{x} = f(x, \lambda)$. In view of our demand that Γ appears generically within the family $\dot{x} = f(x, \lambda)$ we impose the transversality condition

$$\textbf{(T)} \qquad \left. \frac{\partial(\hat{B}(\cdot,\cdot,0), D_1\hat{B}(\cdot,\cdot,0)}{\partial(v, \lambda)} \right|_{(v,\lambda)=(0,0)} \qquad \text{has rank } d^2.$$

Recalling that $(\hat{B}(\cdot,\cdot,0), D_1\hat{B}(\cdot,\cdot,0)) : Y \times \mathbb{R}^m \to \ker\ L^* \times \mathbb{L}(Y, \ker L^*)$ we see that (T) is a maximal rank condition and can only be satisfied if $m \geq d^2 - d + 1$.
Subsequently we restrict ourselves on the case $d = 2$.
In this case the condition (T) coindides with the transversality condition given in [9]. There is shown that (for $d = 2$ and $m = 3$) (T) implies that the set of parameters λ for which the system $\dot{x} = f(x, \lambda)$ has a homoclinic orbit near Γ has a stable singularity.
Now we turn to the investigation of the bifurcation equation (12) under the assumption $d = 2$. We will consider B as a mapping

$$\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2, \quad (\varrho, \sigma, \lambda, \epsilon) \mapsto \mathcal{B}(\varrho, \sigma, \lambda, \epsilon),$$

where $u = \varrho v + \sigma\dot{\gamma}$ (v and $\dot{\gamma}$ are linearly independent elements in $\ker L$). $\mathcal{B}(\varrho, \sigma, \lambda, \epsilon) = (\beta^1(\varrho, \sigma, \lambda, \epsilon), \beta^2(\varrho, \sigma, \lambda, \epsilon))$, where $B(u, \lambda, \epsilon) = \beta^1(\varrho, \sigma, \lambda, \epsilon)w^1 + \beta^2(\varrho, \sigma, \lambda, \epsilon)w^2$ and $\ker L^*$ = span $\{w^1, w^2\}$. Then the bifurcation equation reads as follows

$$\mathcal{B}(\varrho, \sigma, \lambda, \epsilon) = 0. \tag{24}$$

The transformation

$$(\varrho, \sigma, \lambda, \epsilon) \mapsto (\varrho, \sigma - \sigma^*(\varrho, v, \lambda, \epsilon)$$

effects that homoclinic points associated to $\sigma = 0$ are located in Σ.
The homoclinic point $\sigma\dot{\gamma}(0) + \varrho v(0) + y + (\varrho, \sigma, \lambda, \epsilon)(0)$ is related to the solution $(\varrho, \sigma, \lambda, \epsilon)$ of the bifurcation equation (24). Choosing v such that $v(0) \in \Sigma$ yields $y^*(\varrho, 0, \lambda, \epsilon)(0) \in \Sigma$.

Lemma 4 $\mathcal{B}_{\sigma\sigma}(0,0,0,0)=0$.

Proof For $\lambda=0$ and $\epsilon=0$ $\{\gamma(t),\ t\in\mathbb{R}\}$ is a smooth curve of homoclinic points. Associated with this there are smooth functions $\sigma(\cdot)$ and $\varrho(\cdot)$ such that $\mathcal{B}(\varrho(t),\sigma(t),0,0)\equiv 0$ and $\varrho(0)=\sigma(0)=0$. Differentiation with respect to t at $t=0$ yields:

$$\mathcal{B}_{\varrho\varrho}(0)(\dot\varrho(0),\dot\varrho(0))+2\mathcal{B}_{\varrho\sigma}(0)(\dot\varrho(0),\dot\sigma(\cdot))+\mathcal{B}_{\sigma\sigma}(0)(\dot\sigma,\dot\sigma(0))+\mathcal{B}_{\varrho}(0)\ddot\varrho(0)+\mathcal{B}_{\sigma}(0)\ddot\sigma(0)=0.\tag{25}$$

Note that $\mathcal{B}_{\varrho}(0)=\mathcal{B}_{\sigma}(0)=0$. Together with (25) this yields $\mathcal{B}_{\sigma\sigma}(0,0,0,0)(1,1)=0$.

It remains to prove $\dot\varrho(0)=0$ and $\dot\sigma(0)=1$: To find homoclinic points we look for solutions of (1) in the form $x(t)=\gamma(t)+z(t)$ - cf. (7). The homoclinic points are the corresponding points $x(0)$, where $z(0)=\sigma\dot\gamma(0)+\varrho v(0)+y^*(\varrho,\sigma,\lambda,\epsilon)(0)$. Hence for small $|t|$ we have $\gamma(t)=\gamma(0)+\sigma(t)\dot\gamma(0)+\varrho(t)v(0)+y^*(\varrho(t),\sigma(t),0,0)(0)$. From this we get again by differentiating $\dot\gamma(0)=\dot\sigma(0)\dot\gamma(0)+\dot\varrho(0)v(0)+y^*_{\varrho}(0)\dot\varrho(0)+y^*_{\sigma}(0)\dot\sigma(0)$. Due to $y^*_{\varrho}(0)=y^*_{\sigma}(0)=0$ - this is again a consequence of the Liapunov/Schmidt reduction - this reduces to $\dot\gamma(0)=\dot\sigma(0)\dot\gamma(0)+\dot\varrho(0)v(0)$. Since, on the other hand, $\dot\gamma(0)$ and $v(0)$ are linearly independent ($\dot\gamma$ and v are linearly independent solutions of one linear differential equation) we get $\dot\sigma(0)=1$ and $\dot\varrho(0)=0$. ∎

Because of $\hat B(\varrho v,\lambda,\epsilon)=B(\sigma^*(\varrho v,\lambda,\epsilon)\dot\gamma+\varrho v,\lambda,\epsilon)$ and $D_1B(0,0,0)=0$ we get $D_2\hat B(0,0,0)=D_2B(0,0,0)$. A consequence of (T) is rank $D_2\tilde B(0,0,0)=2$ and hence also rank $D_2\mathcal{B}(0,0,0)=2$.

Therefore, without loss of generality, we can solve the bifurcation equation (24) for $\lambda_i=\lambda_i^*(\varrho,\sigma,\lambda_3,\epsilon)$, $i=1,2$, near $(\varrho,\sigma,\lambda,\epsilon)=(0,0,0,0)$. This means nothing else but the set of parameters λ,ϵ for which equation (1) has homoclinic points near $\gamma(0)$ is just the image of Λ,

$$\Lambda:\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^4\quad,\quad(\varrho,\sigma,\mu,\epsilon)\mapsto\begin{pmatrix}\lambda_1^*(\varrho,\sigma,\mu,\epsilon)\\ \lambda_2^*(\varrho,\sigma,\mu,\epsilon)\\ \mu\\ \epsilon\end{pmatrix}.\tag{26}$$

Again $D_1B(0,0,0)=0$ yields

$$D\Lambda(\cdot,\cdot,0,0)\mid_{(\varrho,\sigma)=(0,0)}=0.\tag{27}$$

In view of Theorem 3 we mention that for each sufficiently small $(\varrho,\sigma,\mu,\epsilon)$

$$\begin{aligned}2\ &=\operatorname{rank}\left.\frac{\partial(\lambda_1^*,\lambda_2^*)}{\partial(\varrho,\sigma)}\right|_{(\varrho,\sigma,\mu,\epsilon)}\\ &=\dim\ker D_2B(\varrho v+\sigma\dot\gamma+y^*(\varrho,\sigma,\lambda_1^*,\lambda_2^*,\mu,\epsilon),\lambda_1^*,\lambda_2^*,\mu,\epsilon).\end{aligned}\tag{28}$$

This is an immediate consequence of the non-singularity of $\frac{\partial\mathcal{B}}{\partial(\lambda_1,\lambda_2)}(0,0,0,0)$.

By differentiating $\mathcal{B}(\varrho,\sigma,\lambda_1^*,\lambda_2^*,\mu,\epsilon)\equiv 0$ (with respect to σ) and taking into consideration $\frac{\partial\lambda_i^*}{\partial\sigma}(0,0,0)=0$ (cf. (27)) and $B_{\sigma\sigma}(0,0,0,0)=0$ (cf. Lemma 4) we get

$$\frac{\partial^2(\lambda_1^*,\lambda_2^*)}{\partial\sigma^2}(0,0,0,0)=0.\tag{29}$$

Subsequently we will perform transformations similar to [3] but taking into consideration that here Λ also depends on σ. However, we will do it in a way that Λ for $\sigma = 0$ coincides with that one in [3].
(T) implies that

$$\left.\begin{pmatrix} \frac{\partial^2\lambda_1^*}{\partial\varrho^2} & \frac{\partial^2\lambda_2^*}{\partial\varrho^2} \\ \frac{\partial^2\lambda_1^*}{\partial\varrho\lambda\mu} & \frac{\partial^2\lambda_2^*}{\partial\varrho\partial\mu} \end{pmatrix}\right|_{(\varrho,\sigma,\mu,\epsilon)=(\epsilon,0,0,0)} \quad \text{is non} - \text{singular.} \tag{30}$$

From that we conclude (without loos of generality)

$$\frac{\partial^2\lambda_1^*}{\partial\varrho^2}(0,0,0,0) \neq 0, \quad \frac{\partial^2\lambda_2^*}{\partial\varrho\partial\mu}(0,0,0,0) \neq 0. \tag{31}$$

Taking into consideration also (27) we can solve $\frac{\partial\lambda_1^*}{\partial\varrho}(\varrho,0,\mu,\epsilon) = 0$ for $\varrho = \varrho^*(\mu,\epsilon)$, near $(0,0,0,0)$.
With that we define the transformation T_1 by

$$T_1 : (\varrho,\sigma,\mu,\epsilon) \mapsto (\varrho - \varrho^*(\mu,\epsilon),\sigma,\mu,\epsilon).$$

Then the transformed λ_1^* takes the form

$$\begin{aligned} \lambda_1'(\varrho,\sigma,\mu,\epsilon) &:= \lambda_1^*(T_1^{-1}(\varrho,\sigma,\mu,\epsilon)) \\ &= a_1(\sigma,\mu,\epsilon) + a_2(\sigma,\mu,\epsilon)\sigma\varrho + \varkappa(\varrho,\sigma,\mu,\epsilon)\varrho^2. \end{aligned}$$

This is nothing else but the Taylor expansion (with respect to ϱ) up to second order. The transformation T_1 provides that the coefficient of the first order term vanishes for $\sigma = 0$. Writing $\varkappa$ as $\varkappa(\varrho,\sigma,\mu,\epsilon) = \varkappa_1'(\varrho,0,\mu,\epsilon) + \varkappa_2'(\varrho,\sigma,\mu,\epsilon)\sigma$ we get

$$\lambda_1'(\varrho,\sigma,\mu,\epsilon) = a_1(\sigma,\mu,\epsilon) + a_2'(\varrho,\sigma,\mu,\epsilon)\sigma\varrho + \varkappa_1'(\varrho,0,\mu,\epsilon)\varrho^2. \tag{32}$$

Due to $\frac{\partial^2\lambda_1^*}{\partial\varrho^2}(0,0,0,0) \neq 0$ we get $\varkappa_1'(0,0,0,0) \neq 0$. This gives rise to a second transformation

$$T_2 : (\varrho,\sigma,\mu,\epsilon) \mapsto (\varrho\sqrt{\mid \varkappa_1'(\varrho,0,\mu,\epsilon) \mid},\ \sigma,\mu,\epsilon)$$

and brings λ_1' in the form

$$\begin{aligned} \lambda_1''(\varrho,\sigma,\mu,\epsilon) &= \lambda_1'(T_2^{-1}(\varrho,\sigma,\mu,\epsilon)) \\ &= a_1(\sigma,\mu,\epsilon) + a_2''(\varrho,\sigma,\mu,\epsilon)\sigma\varrho + \underbrace{\operatorname{sgn}(\varkappa_1'(0,0,0,0))}_{=:\delta}\varrho^2 \ . \end{aligned}$$

Simultaneously we define $\lambda_2''(\varrho,\sigma,\mu,\epsilon) = \lambda_2^*((T_2\circ T_1)^{-1}(\varrho,\sigma,\mu,\epsilon))$. Since $\left.\frac{\partial(T_2\circ T_1)}{\partial(\varrho,\sigma)}\right|_{(0,0,0,0)}$ is non-singular we get

$$\operatorname{rank}\frac{\partial(\lambda_1'',\lambda_2'')}{\partial(\varrho,\sigma)} = \operatorname{rank}\left.\frac{\partial(\lambda_1^*,\lambda_2^*)}{\partial(\varrho,\sigma)}\right|_{(\varrho,\sigma,\mu,\epsilon)}. \tag{33}$$

Moreover, differentiating $(\lambda_1'', \lambda_2'')(T_2 \circ T_1(\varrho, \sigma, \mu, \epsilon)) = (\lambda_1^*, \lambda_2^*)(\varrho, \sigma, \mu, \epsilon)$ twice with respect to σ (at zero) provides

$$\left.\frac{\partial^2(\lambda_1'', \lambda_2'')}{\partial\sigma^2}\right|_{(0,0,0)} = \left.\frac{\partial^2(\lambda_1^*, \lambda_2^*)}{\partial\sigma^2}\right|_{(0,0,0)} = 0 \tag{34}$$

cf. (29). For that we exploit that the transformation of ϱ does not depend on σ.

To get rid of the zero order terms of the Taylor expansion of $(\lambda_1'', \lambda_2'')$ with respect to ϱ, σ we perform the following transformation in $(\lambda, \epsilon)-$space:

$$\Phi_1 : (\lambda_1, \lambda_2, \lambda_3, \epsilon) \mapsto \begin{pmatrix} \delta(\lambda_1 - \lambda_1''(0,0,\lambda_3,\epsilon) \\ \delta(\lambda_2 - \lambda_2''(0,0,\lambda_3,\epsilon) \\ \lambda_3 \\ \epsilon \end{pmatrix}. \tag{35}$$

Hence

$$\begin{aligned}\Phi_1(\lambda_1''(\cdots), \lambda_2''(\cdots), \mu, \epsilon) = \; & (\underbrace{a_1'(\sigma,\mu,\epsilon)\sigma + a_2''(\varrho,\sigma,\mu,\epsilon)\sigma\varrho + \varrho^2}_{=:\lambda_1'''(\varrho,\sigma,\mu,\epsilon)}, \\ & \underbrace{b_1(\mu,\epsilon)\varrho + b_2(\varrho,\mu,\epsilon)\varrho^2 + b_3(\varrho,\sigma,\mu,\epsilon)\sigma}_{=:\lambda_2'''(\varrho,\sigma,\mu,\epsilon)}, \mu, \epsilon).\end{aligned} \tag{36}$$

We obtain the structure of λ_2''' by calculating the Taylor expansion with respect to σ (up to first order) and then expanding the zero order term with respect to ϱ (up to second order). (34) provides immediately

$$\frac{\partial}{\partial\sigma}a_1'(0,0,0) = \frac{\partial}{\partial\sigma}b_3(0,0,0,0) = 0 \tag{37}$$

Of course also

$$a_1'(0,0,0) = b_3(0,0,0,0) = 0 \tag{38}$$

holds true. This is due to (33) and $\frac{\partial\lambda_i^*}{\partial\sigma}(0,0,0) = 0$.
From (35) and (36) we see that $\lambda_i'''(\varrho,\sigma,\mu,\epsilon) = \delta(\lambda_i''(\varrho,\sigma,\mu,\epsilon) - \lambda_i''(0,0,\mu,\epsilon))$. Hence

$$\operatorname{rank}\frac{\partial(\lambda_1''', \lambda_2''')}{\partial(\varrho,\sigma)} = \operatorname{rank}\frac{\partial(\lambda_1'', \lambda_2'')}{\partial(\varrho,\sigma)}. \tag{39}$$

(33) yields

$$b_1(0,0) = 0, \tag{40}$$

and as a consequence of (30) and (31) we get

$$\frac{\partial b_1}{\partial\mu}(0,0) \neq 0. \tag{41}$$

b_2 can be rewritten in $b_2(\varrho,\mu,\epsilon) = b_{2e}(\varrho^2,\mu,\epsilon) + \varrho b_{2o}(\varrho^2,\mu,\epsilon)$, where b_{2e} is the even part and ϱb_{2o} is the odd part of b_2 with respect to ϱ. With that we define Φ_2 as follows:

$$\Phi_2 : (\lambda_1,\lambda_2,\lambda_3,\epsilon) \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 - \lambda_1 b_{2e}(\lambda_1,\lambda_3,\epsilon) \\ b_1(\lambda_3,\epsilon) + \lambda_1 b_{2o}(\lambda_1,\lambda_3,\epsilon) \\ \epsilon \end{pmatrix}.$$

Let $(\lambda_1^{IV}(\varrho,\sigma,\mu,\epsilon), \lambda_2^{IV}(\cdots), \lambda_3^{IV}(\cdots),\epsilon) := \Phi_2(\lambda_1'''(\varrho,\sigma,\mu,\epsilon),\lambda_2'''(\cdots),\mu,\epsilon)$. Obviously we get $\lambda_1^{IV} = \lambda_1'''$. Further

$$\begin{aligned} \lambda_2^{IV}((\varrho,\sigma,\mu,\epsilon)) &= b_1(\mu,\epsilon)\varrho + (b_{2e}(\varrho^2,\mu,\epsilon) + \varrho b_{2o}(\varrho^2,\mu,\epsilon))\varrho^2 + \sigma b_3(\cdots) \\ &\quad -(a_1'\sigma + a_2''\sigma\varrho + \varrho^2) b_{2e}(a_1'\sigma + a_2''\sigma\varrho + \varrho^2,\mu,\epsilon) \\ &= b_1(\mu,\epsilon)\varrho + \varrho^3 b_{2o}(\varrho^2,\mu,\epsilon) + \sigma b_3^{IV}(\cdots). \end{aligned}$$

This can be obtained by expanding $b_{2e}(a_1'\sigma + a_2''\sigma\varrho + \varrho^2,\mu,\epsilon)$ at (ϱ^2,μ,ϵ). (37) and (38) ensure that again

$$b_3^{IV}(0,0,0) = \frac{\partial}{\partial\sigma} b_3^{IV}(0,0,0) = 0. \tag{42}$$

This means nothing else but

$$\frac{\partial^2 \lambda_2^{IV}(0,0,0,0)}{\partial\sigma^2} = 0. \tag{43}$$

Similar we get $\lambda_3^{IV}(\varrho,\sigma,\mu,\epsilon) = b_1(\mu,\epsilon) + \varrho^2 b_{2o}(\varrho^2,\mu,\epsilon) + \sigma c_1(\varrho,\sigma,\mu,\epsilon)$. Again (37) and (38) ensure that

$$c_1(0,0,0,0) = \frac{\partial}{\partial\sigma} c_1(0,0,0,0) = 0. \tag{44}$$

Immediately from the definition of $\lambda_i^{IV},\ i = 1,2$, we get

$$\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} = \begin{pmatrix} \frac{\partial}{\partial\varrho}\lambda_1''' & \frac{\partial}{\partial\sigma}\lambda_1''' \\ \frac{\partial}{\partial\varrho}\lambda_2''' - \frac{\partial}{\partial\varrho}\lambda_1'''(b_{2e} + \lambda_1\frac{\partial}{\partial\lambda_1}b_{2e}) & \frac{\partial}{\partial\sigma}\lambda_2''' - \frac{\partial}{\partial\sigma}\lambda_1(b_{2e} + \lambda_1\frac{\partial}{\partial\lambda_1}b_{2e}) \end{pmatrix}.$$

Therefore

$$\operatorname{rank}\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} = \operatorname{rank}\frac{\partial(\lambda_1''',\lambda_2''')}{\partial(\varrho,\sigma)}. \tag{45}$$

After the final transformation

$$T_3 : (\varrho,\sigma,\mu,\epsilon) \mapsto (\varrho,\sigma,\underbrace{b_1(\mu,\epsilon) + \varrho^2 b_{2o}(\varrho^2,\mu,\epsilon)}_{=:\mu(\varrho,\mu,\epsilon)},\epsilon)$$

Λ takes the form

$$(\varrho,\sigma,\mu,\epsilon) \mapsto \begin{pmatrix} \lambda_1^V(\varrho,\sigma,\mu,\epsilon) \\ \lambda_2^V(\varrho,\sigma,\mu,\epsilon) \\ \lambda_3^V(\varrho,\sigma,\mu,\epsilon) \\ \epsilon \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned} \lambda_1^V(\varrho,\sigma,\mu,\epsilon) &= \varrho^2 + \sigma a(\varrho,\sigma,\mu,\epsilon), & \text{(47-a)} \\ \lambda_2^V(\varrho,\sigma,\mu,\epsilon) &= \varrho\mu + \sigma b(\varrho,\sigma,\mu,\epsilon), & \text{(47-b)} \\ \lambda_3^V(\varrho,\sigma,\mu,\epsilon) &= \mu + \sigma c(\varrho,\sigma,\mu,\epsilon). & \text{(47-c)} \end{aligned}$$

(41) ensures that T_3 is indeed a transformation.
Again

$$a(0,0,0,0) = 0,\ b(0,0,0,0) = 0,\ \frac{\partial}{\partial\sigma}a(0,0,0,0) = 0,\ \frac{\partial}{\partial\sigma}b(0,0,0,0) = 0. \tag{48}$$

It is obvious that for $\sigma = 0$ the set of parameters for which (1) has homoclinic points - or in other words the homoclinic points in a hyperplan Σ orthogonal to $\dot\gamma(0)$ - forms a family of Whitney umbrellas. This is the main result obtained in [3]. But to characterize the homoclinic points by the intersection number $D_{\lambda,\epsilon,u}$ (even if we are only interested in $D_{\lambda,\epsilon,u}$ for homoclinic points in Σ) we have to take into considerations also directions transversal to Σ.
Starting with (28) and taking the way along (33), (39) and (45) we get finally

$$D_{\varrho,\sigma,\mu,\epsilon} = 2 - \operatorname{rank}\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)}. \tag{49}$$

We will show that for $\sigma = 0$ $\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)}$ can be replaced by $\frac{\partial(\lambda_1^{V},\lambda_2^{V})}{\partial(\varrho,\sigma)}$ in formula (49).
Exploiting (47) we get

$$\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0} = \begin{pmatrix} 2\varrho & a(\varrho,0,\mu,\epsilon) \\ \mu & b(\varrho,0,\mu,\epsilon) \end{pmatrix}. \tag{50}$$

Before starting with the investigation of $\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0}$ we make assumptions regarding the periodic perturbation $g(t,x,\lambda,\epsilon)$:

Let

$$\begin{aligned} a(\varrho,0,\mu,\epsilon) &= \bar a_o(\varrho,\mu) + \bar a_1(\varrho,\mu)\epsilon + \bar a_2(\varrho,\mu,\epsilon)\epsilon^2 \\ b(\varrho,0,\mu,\epsilon) &= \bar b_o(\varrho,\mu) + \bar b_1(\varrho,\mu)\epsilon + \bar b_2(\varrho,\mu,\epsilon)\epsilon^2. \end{aligned}$$

Using this notation we suppose:

(A1) $\begin{pmatrix} a(0,0,0,\epsilon) \\ b(0,0,0,\epsilon) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

(A2) For $M(\varrho,\mu) := 2\varrho\bar{b}_1(\varrho,\mu) - \mu\bar{a}_1(\varrho,\mu)$; we suppose $DM(0,0) = 0$ and $D^2M(0,0)$ to be definite.

$(A2)$ is sufficient for the definiteness of M in a neighborhood of $(\varrho,\mu) = (0,0)$. $(A2)$ includes $\bar{a}_1(0,0) = \bar{b}_1(0,0) = 0$, which means nothing else but $\frac{\partial}{\partial\sigma\partial\epsilon}(\lambda_1^V,\lambda_2^V)|_{(0,0,0,0)} = 0$ or in other words $\frac{\partial^2}{\partial\sigma\partial\epsilon}B(0,0,0,0) = 0$. The definiteness of $D^2M(0,0)$ implies

$$\frac{\partial}{\partial\varrho}\bar{b}_1(0,0) \neq 0. \tag{51}$$

Theorem 5 *Assume* $(A1),(A2)$ *and* (T).
For $\epsilon = 0$ *is* $D_{\varrho,0,\mu,0} \geq 1$, *where* $D_{\varrho,0,\mu,0} = 2$ *if and only if* $\varrho = \mu = 0$. *For* $\epsilon \neq 0$ *is* $D_{\varrho,0,\mu,\epsilon} \leq 1$, *where only* $D_{0,0,0,\epsilon} = 1$.

Proof First we consider the relation between $\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0}$ and $\left.\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)}\right|_{\sigma=0}$.

Recalling $\lambda_i^V(T_3(\varrho,\sigma,\mu,\epsilon)) = \lambda_i^{IV}(\varrho,\sigma,\mu,\epsilon)$ we see $\frac{\partial}{\partial\varrho(\sigma)}\lambda_i^{IV} = \frac{\partial}{\partial\varrho(\sigma)}\lambda_i^V + \frac{\partial}{\partial\mu}\lambda_i^V\frac{\partial}{\partial\varrho(\sigma)}\mu(\varrho,\mu,\epsilon)$, for $i = 1,2$. Hence for $\sigma = 0$ we get

$$\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} = \frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)} + \begin{pmatrix} 0 & 0 \\ o(\varrho^2) & 0 \end{pmatrix}. \tag{52}$$

From that we see:
Let $\epsilon = 0$: A homoclinic point associated to $(\varrho,\sigma = 0,\mu)$ is lying on a homoclinic orbit of $\dot{x} = f(x,\lambda)$. Hence rank $\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} \leq 1$.
If $(\varrho,\mu) \neq (0,0)$ than rank $\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0} = 1$ and hence rank $\left.\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)}\right|_{\sigma=0} = 1$.
On the other hand for $\varrho = \mu = 0$ we have rank $\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0} = 2$ due to our assumption that Γ is degenerate.
Now let $\epsilon \neq 0$: From our previous considerations we see that

$$\det\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0} = M(\varrho,\mu)\epsilon + 0(\epsilon^2).$$

Then $(A2)$ implies that for $(\varrho,\mu) \neq (0,0)$ det $\left.\frac{\partial(\lambda_1^V,\lambda_2^V)}{\partial(\varrho,\sigma)}\right|_{\sigma=0} \neq 0$ for sufficiently small ϵ. Then expressions (51) and (52) yield that also det $\left.\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)}\right|_{\sigma=0} \neq 0$.
If remains to consider the homoclinic points associated to $(\varrho,\mu,\epsilon) = (0,0,\epsilon \neq 0)$. These are just the homoclinic points corresponding to parameters defining the singularity of the associated Whitney umbrella. (52) shows that in this case det $\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} = 0$.
But $(A1)$ ensures that rank $\frac{\partial(\lambda_1^{IV},\lambda_2^{IV})}{\partial(\varrho,\sigma)} = 1$. ∎

This result confirms the intuitive picture one can draw: Fix $\epsilon \neq 0$ and put $\sigma = 0$. $\{(\lambda_1, \lambda_2, \lambda_3, \epsilon) = (\varrho^2, \mu\varrho, \mu, \epsilon), (\varrho, \mu) \in U(0,0)\}$ - the set of parameters for which homoclinic points exist - forms a Whitney umbrella. Along the line of selfintersection - $\{(\varrho, 0, 0, \epsilon)\}$ there exist two different homoclinic points associated to ϱ and $-\varrho$. If ϱ tends to zero these homoclinic points become closer and closer and merge finally at $\varrho = 0$ to a non-transversal homoclinic point.
But there can also rise non-transversal homoclinic points by tending ϵ to zero. For $\epsilon = 0$ there is a layered bifurcation - for $\epsilon = 0$ there is a curve of homoclinic points coinciding with the homoclinic orbit of the associated equation $\dot{x} = f(x, \lambda)$.
Do λ and ϵ tend to zero both types of bifurcations take place simultaneously. This effects the intersection number $D_{0,0,0,0} = 2$.

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